Direct Solution of the "Three-Moments Equation"

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ONE hundred years ago Clapeyron noticed that the bending moments at three consecutive supports are connected by an invariable relation. When the spans are equal, the corresponding equation and the most useful boundary conditions can be written in the form

$$M_{n+1} + 4M_n + M_{n-1} = -G_n \qquad (1)$$

$$M_0 = M_N = 0 \qquad n = 0, 1, \ldots, N \qquad (2)$$

where M_n and G_n are moments and loading terms at consecutive supports. With help of the symmetrical operator

$$A_{nm} = \tfrac{1}{2}[|n - m| - (n + m) + (2nm/N)] \qquad (3)$$

(introduced by the author), it is possible to express the exact solution of the boundary problem (1) and (2) in the form

$$M = (\mathbf{I} - 6\mathbf{A})^{-1}\mathbf{A}G \qquad (4)$$

The operator $\mathbf{A}$ is nonsingular; thus

$$(\mathbf{I} - 6\mathbf{A})^{-1} = \sum_{\alpha=0}^{\infty} 6^{\alpha}\mathbf{A}^{\alpha} \qquad (5)$$

and

$$M = -\frac{1}{6}\left(\mathbf{I} - \sum_{\alpha=0}^{\infty} 6^{\alpha}\mathbf{A}^{\alpha}\right)G \qquad (6)$$

But for symmetrical tensor $\mathbf{A}$, one can write

$$\mathbf{A} = (1/\lambda_i)\mathbf{P}_{(i)}$$

and

$$\mathbf{A}^{\alpha} = [1/(\lambda_i)^{\alpha}]\mathbf{P}_{(i)} \qquad i = 0, 1, \ldots, N \qquad (7)$$

where λ_i are eigenvectors, roots of the equation

$$|\mathbf{I} - \lambda\mathbf{A}| = 0$$

and

$$P_{(i)n,m} = C_{(i)n}C_{(i)m}$$

are projection tensors associated with orthogonal directions $C_{(i)}$ called normalized eigenvectors of $\mathbf{A}$.

Hence, introducing (7) into (5) yields

$$\sum_{\alpha=0}^{\infty} 6^{\alpha}\mathbf{A}^{\alpha} = \sum_{\alpha=0}^{\infty}\left(\frac{6}{\lambda_i}\right)^{\alpha}\mathbf{P}_{(i)}$$

When $(\mathbf{I} - 6\mathbf{A})^{-1}$ exist, then the series $\sum_{\alpha=0}^{\infty} (6/\lambda_i)^{\alpha}$ is convergent to $\lambda_i/(\lambda_i - 6)$ and

$$\sum_{\alpha=0}^{\infty} 6^{\alpha}\mathbf{A}^{\alpha} = \sum_{i=0}^{N} \frac{\lambda_i}{\lambda_i - 6}\mathbf{P}_{(i)}$$

The solution of the "three-moments equation" is therefore

$$M = -\frac{1}{6}\left(\mathbf{I} - \sum_{i=0}^{N} \frac{\lambda_i}{\lambda_i - 6}\mathbf{P}_{(i)}\right)G \qquad (8)$$

Because of the boundary conditions (2), the first and last member of the series in (8) can be neglected, and the solution can be written as follows:

$$M = -\frac{1}{6}\left(\mathbf{I} - \sum_{i=1}^{N-1} \frac{\lambda_i}{\lambda_i - 6}\mathbf{P}_{(i)}\right)G \qquad (9)$$

When, for example $N = 3$, then $\lambda_1 = 1$, $\lambda_2 = 3$, and

$$C_{(1)} = \begin{vmatrix} \dfrac{1}{2^{1/2}} \\ \dfrac{1}{2^{1/2}} \end{vmatrix} \qquad C_{(2)} = \begin{vmatrix} \dfrac{1}{2^{1/2}} \\ -\dfrac{1}{2^{1/2}} \end{vmatrix}$$

thus Eq. (8) takes the form

$$\begin{vmatrix} M_1 \\ M_2 \end{vmatrix} = -\frac{1}{6}\left(\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \frac{1}{5}\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} - \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}\right)\begin{vmatrix} G_1 \\ G_2 \end{vmatrix} \qquad (10)$$

Assuming uniform load on each span and length of the span equal l, the load terms are $G_1 = G_2 = \frac{1}{2}pl^2$, and one therefore has

$$\begin{vmatrix} M_1 \\ M_2 \end{vmatrix} = -\frac{1}{6}\begin{vmatrix} \frac{8}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{8}{5} \end{vmatrix}\begin{vmatrix} (pl^2/2) \\ (pl^2/2) \end{vmatrix}$$

in agreement with results of the theory of elasticity.

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A Similar Solution of the Turbulent, Free-Convection, Boundary Layer Problem for an Electrically Conducting Fluid in the Presence of a Magnetic Field

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THE free-convection boundary layer problem for an electrically conducting fluid in the presence of a magnetic field has been considered in Refs. 1–4 for the case of laminar flow. The purpose of this paper is to investigate the analogous problem for turbulent flow.

The physical model considered below is similar to that which was employed by Sparrow.[2] A magnetic field is impressed across a vertical plate that is kept at a constant temperature T_w in an electrically conducting fluid of ambient temperature T_e and conductivity σ.

If the usual[3] MHD, free-convection, boundary layer simplifications are adopted, the integrated momentum and energy equations can be written as follows:[3]

$$\frac{d}{dx}\int_0^{\delta} u^2 dy = g\beta\int_0^{\delta} \theta dy - \frac{\tau_w}{\rho} - \frac{1}{\rho}\int_0^{\delta} \sigma B_0 u dy \qquad (1)$$

$$q_w = g\rho C_p \frac{d}{dx}\int_0^{\delta} u\theta dy \qquad (2)$$

The notation used here is as usually employed in free-convection boundary layer analysis.

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